Digital

29/2/2016 الاثنين د. فرضة محاضرة 3

* Steady State error (S.S.e)

$e(t) \longrightarrow$ S.S.e: the error at $t \to \infty$

$$S.S.e = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s)$$

$$= \lim_{z\to 1} (1-z^{-1}) E(z) = \lim_{z\to 1} (z-1) E(z)$$

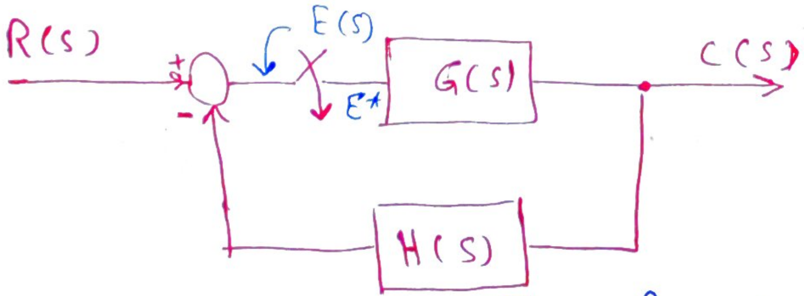
Cont. time

1) $r(t) = 1$ (Unit Step) $\longleftrightarrow$ $S.S.e = \dfrac{1}{1+K_p}$ $\Rightarrow K_p = \lim_{s\to 0} GH(s)$ [Position error constant]

2) $r(t) = t$ (Unit ramp) $\Rightarrow S.S.e = \dfrac{1}{K_v}$ $\Rightarrow K_v = \lim_{s\to 0} sGH(s)$ [Velocity error constant]

3) $r(t) = t^2/2 \Rightarrow S.S.e = \dfrac{1}{K_a}$ $\Rightarrow K_a = \lim_{s\to 0} s^2 GH(s)$ [acceleration error constant]

* Steady-State error in discrete-time System



The general form for discrete-time feedback

$$S.S.e = \lim_{s\to 0} sE(s) = \lim_{z\to 1} (z-1)E(z)$$

? in terms of R(z)

$C(s) = GE^*$

$E(s) = R(s) - GHE^* \longrightarrow$ Staring $E^* = R^* - \overline{GH}^* E^*$

$(1 + \overline{GH}^*) E^* = R^*$

$$E^*(s) = \frac{R^*(s)}{1 + \overline{GH(s)}^*} \Longrightarrow E(z) = \frac{R(z)}{1 + \overline{GH}(z)}$$

$$\therefore \; s.s.e = \lim_{z \to 1} (z-1) \frac{R(z)}{1+\overline{GH}(z)}$$

$\Downarrow$ depend on

① System i/p $\Rightarrow R(z)$

② Digital O.L.T.F $= \overline{GH}(z)$ or system type

The general form for Digital O.L.T.F

$$\overline{GH}(z) = \frac{N(z)}{(z-1)^n D(z)}$$

where $n \rightarrow$ system type

OR

$$\overline{GH}(z) = \frac{N(z)}{(1-z^{-1})^n D(z)}$$

wher $n \longrightarrow$ system type

* for unit step input

$r(t) = 1 \Rightarrow r(kT) = 1 \Rightarrow \boxed{R(z) = \frac{z}{z-1}}$

$$S.S.e = \lim_{z \to 1} (z-1) \frac{R(z)}{1+\overline{GH}(z)}$$

$$= \lim_{z \to 1} (\cancel{z-1}) \frac{z}{(\cancel{z-1})(1+\overline{GH}(z))} = \lim_{z \to 1} \frac{z}{1+\overline{GH}(z)}$$

$$S.S.e = \frac{1}{1+\lim_{z \to} \overline{GH}(z)}$$

$$= \frac{1}{1+K_p}$$

$K_p = \lim_{z \to 1} \overline{GH}(z) \equiv$ Position Error Constant

$\Longrightarrow$ continue

* For unit ramp input

$$r(t) = t \Rightarrow r(KT) = KT \Rightarrow \boxed{R(z) = \frac{Tz}{(z-1)^2}}$$

$$S.S.e = \lim_{z\to 1} (z-1) \frac{R(z)}{1+\overline{GH}(z)}$$

$$= \lim_{z\to 1} \cancel{(z-1)} \frac{Tz}{(z-1)^2(1+\overline{GH}(z))}$$

$$S.S.e = \lim_{z\to 1} \frac{Tz}{(z-1)+(z-1)\overline{GH}(z)}$$

$$= \frac{T}{0 + \lim_{z\to 1}(z-1)\overline{GH}(z)}.$$

$$= \frac{1}{\frac{1}{T}\lim_{z\to 1}(z-1)\overline{GH}(z)} = \frac{1}{K_v}$$

$$K_v = \frac{1}{T}\lim_{z\to 1}(z-1)\overline{GH}(z) \equiv \text{Velocity error constant}$$

* For acceleration input

$$r(t) = t^2 \quad ; \quad t \xrightarrow{z.T} \frac{Tz}{(z-1)^2}$$

$$R(z) = Z\Big(t.\underset{\Downarrow\atop t}{f(t)}\Big|_{t=kT}\Big) = -Tz\frac{d\,f(z)}{dz}$$

$$= Z(t.t) = -Tz\frac{d}{dz}\left(\frac{Tz}{(z-1)^2}\right)$$

$$= -T^2z\left[\frac{(z-1)^2 - z(2)\cancel{(z-1)}}{(z-1)^{\cancel{4}\,3}}\right]$$

$$= -T^2z\left[\frac{(z-1)-2z}{(z-1)^3}\right]$$

$$\boxed{R(z) = \frac{T^2\,z(z+1)}{(z-1)^3}}$$

$\Longrightarrow$ Continue

So, $t^2 \xrightarrow{Z.T} \dfrac{T^2 \, z(z+1)}{(z-1)^3}$

$$r(t) = t^2/2 \Rightarrow R(z) = \frac{T^2/2 \; z(z+1)}{(z-1)^3}$$

$$S.S.e = \lim_{z \to 1} (z-1) \frac{R(z)}{1 + \overline{GH}(z)}$$

$$S.S.e = \lim_{z \to 1} \cancel{(z-1)} \frac{T^2/2 \; z(z+1)}{(z-1)^{\cancel{3}\,2} (1 + \overline{GH}(z))}$$

$$= \lim_{z \to 1} \frac{T^2/2 \; z(z+1)}{(z-1)^2 + (z-1)^2 \, \overline{GH}(z)}$$

$$= \frac{T^2}{0 + \lim_{z \to 1} (z-1)^2 \, \overline{GH}(z)}$$

$$S.S.e = \frac{1}{1/T^2 \lim_{z \to 1} (z-1)^2 \, \overline{GH}(z)} = \frac{1}{K_a}$$

$$K_a = \frac{1}{T^2} \lim_{z \to 1} (z-1)^2 \, \overline{GH}(z) = \text{acceleration constant}$$

* Summary :

① $r(t)\big|_{t=KT} = 1 \longrightarrow S.S.e = \dfrac{1}{1 + K_p} \Longleftrightarrow K_p = \lim_{z \to 1} \overline{GH}(z)$

② $r(t) = t\big|_{t=KT} \longrightarrow S.S.e = \dfrac{1}{K_v} \Longleftrightarrow K_v = \dfrac{1}{T} \lim_{z \to 1} (z-1) \, \overline{GH}(z)$

③ $r(t) = t^2/2\big|_{t=KT} \longrightarrow S.S.e = \dfrac{1}{K_a} \Longleftrightarrow K_a = \dfrac{1}{T^2} \lim_{z \to 1} (z-1)^2 \, \overline{GH}(z)$

type \ I/P	unit step $r(t)=1$	unit ramp $r(t)=t$	acceleration i/p $r(t)=t^2/2$
0	$\frac{1}{1+K_p}$	∞	∞
1	0	$1/K_v$	∞
2	0	0	$1/K_a$

Ex1:-

① $\overline{GH}(z) = \dfrac{-10}{(z-1)(1-z)(z^2-1.4z+1)} = \dfrac{10}{(z-1)^2(z^2-1.4z+1)}$

② $\overline{GH}(z) = \dfrac{(z+1)}{(z-1)(z-2)}$

③ $\overline{GH}(z) = \dfrac{z+0.5}{(1-z)^2(z^2-2)}$

Find: (i) System type and order

(ii) error constants

(iii) s.s.e for $r(t)=1$; $r(t)=t$; $r(t)=2+3t+4t^2$

$\boxed{1}$ type = 2 & order = 4 (4th order system)

$$K_p = \lim_{z\to1} \overline{GH}(z) = \infty$$

$$K_v = \frac{1}{T} \lim (z-1)\, \overline{GH}(z) \quad (\text{assume } T = 1\,s)$$

$$= \infty$$

$$K_a = \frac{1}{T^2} \lim (z-1)^2\, \overline{GH}(z) = \frac{1}{1}\, \frac{10}{1-1.4+1} = \frac{10}{0.6}$$

⟹ continue

$r(t) = 1 \longrightarrow s.s.e = \frac{1}{1+K_p} = \frac{1}{\infty} = 0$

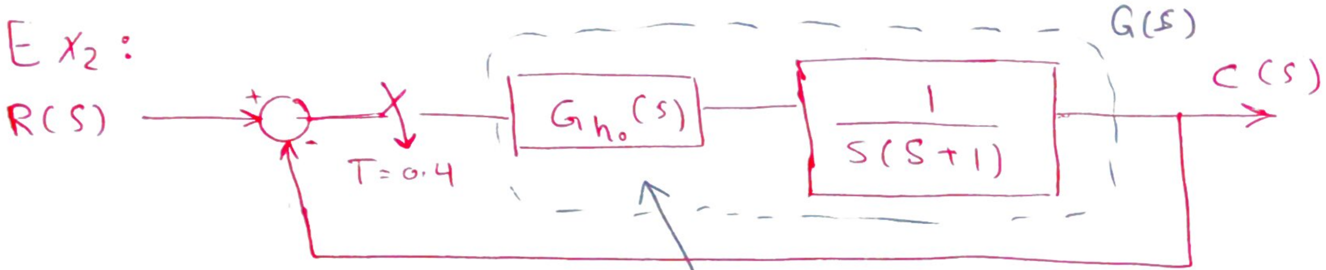
$r(t) = t \longrightarrow s.s.e = \frac{1}{K_v} = \frac{1}{\infty} = 0$

$r(t) = 2 + 3t + 4t^2 \longrightarrow s.s.e = 2\left(\cancel{\frac{1}{1+K_p}}\right)_0 + 3\left(\cancel{\frac{1}{K_v}}\right)_0 + 8\left(\frac{1}{K_a}\right)$

$4t^2 \rightarrow \frac{8t^2}{2}$

$\therefore s.s.e = 8\left(\frac{0.6}{10}\right) = 0.48$

Ex₂ :



$Z.O.H = \frac{1-e^{-TS}}{S}$

Find

1. O.L. pulse T.F.
2. C.L. pulse .T.F.
3. type and order of the system
4. S.S.e for unit step and unit ramp inputs
5. Steady state value of the o/p for unit Step i/p
6. find the first 3 terms of o/p response $y(0), y(0.4), y(0.8)$ for unit Step i/p

① $O.L.T.F = \overline{GH}(z) \Longrightarrow G(z)$

$C.L.T.F = \frac{G(z)}{1+\underbrace{\overline{GH}(z)}_{O.L.T.F}} \Longrightarrow \frac{G(z)}{1+G(z)}$

$O.L.T.F = G(z) = \mathcal{Z}\left[\frac{1-e^{-TS}}{S^2(S+1)}\right]$

$= (1-z^{-1})\,\mathcal{Z}\left[\frac{A}{S^2} + \frac{B}{S} + \frac{C}{S+1}\right]$

$\boxed{A=1}, \boxed{C=1}, \; B \,\|\, \frac{1}{2} = A + B + \frac{C}{2} = 1 + B + \frac{1}{2} \Rightarrow \boxed{B=-1}$

$\Longrightarrow$ Continue

$$O.L.T.F = (1-z^{-1})\, \mathcal{Z}\left[\underset{\uparrow\, 1/s^2}{t} + \underset{\uparrow\, 1/s}{(-1)} + \underset{\uparrow\, 1/(s+1)}{e^{-t}}\right], \quad t = kT$$

$$= (1-z^{-1}) \left[\frac{Tz}{(z-1)^2} - \frac{z}{(z-1)} + \frac{z}{z-e^{-T}}\right]$$

$$= \left(\frac{z-1}{z}\right) \left[\frac{0.4z}{(z-1)^2} - \frac{z}{(z-1)} + \frac{z}{z-0.67}\right]$$

$e^{-T} = e^{-0.4} = 0.67$

$$= \frac{0.4}{(z-1)} - 1 + \frac{z-1}{z-0.67}$$

$$= \frac{0.4(z-0.67) - (z-1)(z-0.67) + (z-1)^2}{(z-1)\quad (z-0.67)(z-1)}$$

$$\boxed{G(z) = \frac{0.07z + 0.062}{(z-1)(z-0.67)}}$$

open loop pulse T.F.
(O.L. Digital T.F.)

note: in general, O.L.T.F. = $GH(z)$; since $H(z) = 1$

$\therefore$ O.L.T.F. = $G(z)$

② $$C.L.T.F = \frac{G(z)}{1+\overline{GH}(z)} = \frac{G(z)}{1+G(z)}$$

$$= \frac{\dfrac{0.07z + 0.062}{(z-1)(z-0.67)}}{1 + \dfrac{0.07z+0.062}{(z-1)(z-0.67)}} = \frac{0.07z + 0.062}{(z-1)(z-0.67) + 0.07z + 0.062}$$

⟹ continue

③ type : 1 | based on O.L.T.F.

order : 2 "second order system" | based on O.L.T.F.

④ s.s.e for unit step and unit ramp

$$K_p = \lim_{z \to 1} GH(z) = \lim_{z \to 1} G(z) = \infty$$

$$s.s.e \Big|_{\substack{r(t)=1 \\ \text{unit step}}} = \frac{1}{1+K_p} = \frac{1}{\infty} = 0$$

$$K_v = \frac{1}{T} \lim_{z \to} (z-1)\, \overline{GH}(z)$$

$$= \frac{1}{0.4} \lim_{z \to 1} (z-1)\, G(z) = \frac{1}{0.4} \left[\frac{0.07 + 0.062}{1 - 0.67}\right] = 1$$

$$\Rightarrow s.s.e = \frac{1}{K_v} = 1$$

⑤ $C(\infty) \equiv$ final value $\equiv$ steady state value of output

$$= \lim_{z \to 1} (z-1)\, C(z)$$

$$C.L.T.F = \frac{C(z)}{R(z)} = \frac{G(z)}{1 + GH(z)} = \left(\frac{0.07z + 0.062}{z^2 - 1.6z + 0.732}\right)$$

$$C(z) = T.F * R(z)$$

$$= \left[\frac{0.07z + 0.062}{z^2 - 1.6z + 0.732}\right] R(z)$$

for unit step i/p $\left(R(z) = \frac{z}{z-1}\right)$

$$C(z) = \frac{(0.07z + 0.062)\, z}{(z-1)(z^2 - 1.6z + 0.732)}$$

$$c(\infty) = \lim_{z \to 1} \frac{(0.07z + 0.062)\,z}{z^2 - 1.6z + 0.732}$$

$$= \frac{0.07 + 0.062}{1 - 1.6 + 0.732} = 1$$

Another method to get $c(\infty)$

$$e(t) = r(t) - y(t)$$

$$\underbrace{e(\infty)}_{s.s.e} = r(\infty) - \underbrace{y(\infty)}_{\text{steady state o/p}}$$

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$$y(\infty) = \underbrace{r(\infty)}_{1} - \underbrace{e(\infty)}_{0} \quad ; \text{ for unit step input}$$

$$\therefore \boxed{y(\infty) = 1 = c(\infty)}$$

⑥ find $C(0)$; $c(0.4)$; $C(0.8)$

from requirement ⑤ we have

$$C(z) = (T.F)\,R(z)$$

$$= \left[\frac{0.07z + 0.062}{z^2 - 1.6z + 0.732}\right]\left(\frac{z}{z-1}\right)$$

$$= \frac{0.07z^2 + 0.062z}{z^3 - 2.6z^2 + 2.33z - 0.732}$$

we use long devision to obtain

first 3 terms

$$
\begin{array}{r|l}
 & 0.07z^{-1} \\
\hline
z^3-2.6z^2+2.33z-0.732 & 0.07z^2+0.062z \\
 & \overset{-}{0.07z^2} \pm 0.182z \mp 0.1631 \pm 0.0512z^{-1} \\
\hline
 & 0.244z-0.163+0.512z^{-1}
\end{array}
$$

$$C(z) = 0.07z^{-1} + 0.244z^{-2} + \cdots$$

$$= \underset{y_0}{C(0)} + C(T)z^{-1} + C(2T)z^{-2} + \cdots$$

$$= C(0) + C(0.4)z^{-1} + C(0.8)z^{-2} + \cdots$$

$C(0) = 0$

$C(T) = C(0.4) = 0.07$

$C(2T) = C(0.8) = 0.244$

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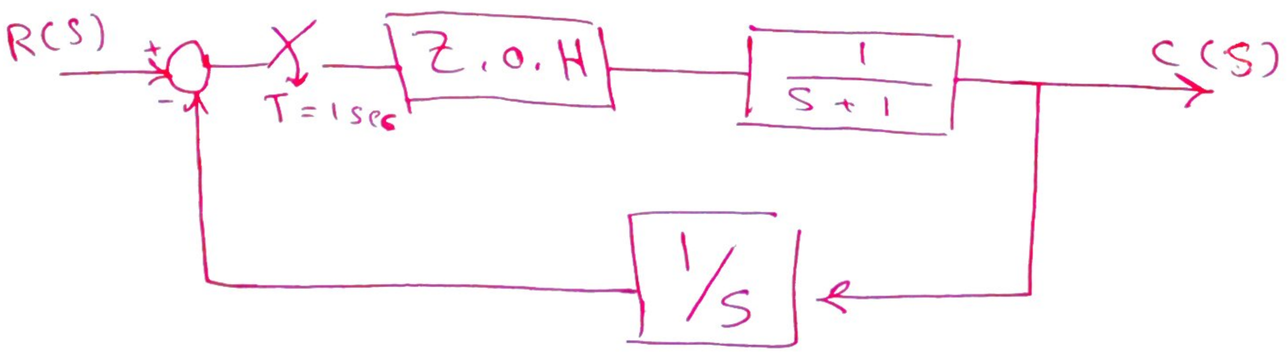


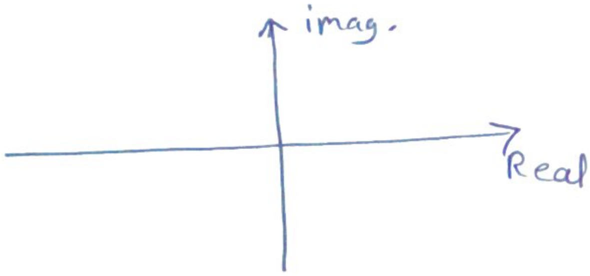
Find the Unit-Step response and discuss system Stability

Relation between Z-domain and S-domain

$Z = e^{TS}$

for $S = \alpha + j\omega$

Real ↙ ↘ img



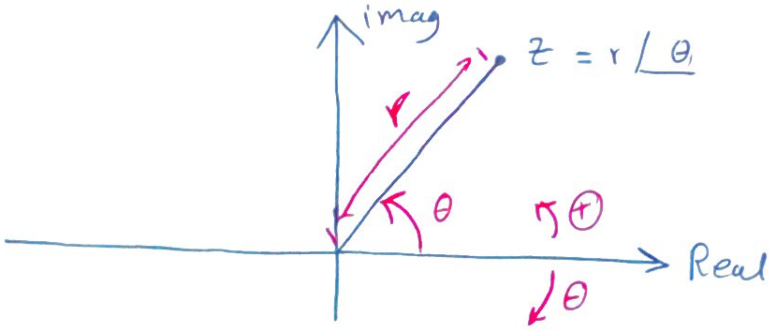
$Z = e^{T(\alpha + j\omega)}$

$Z = e^{\alpha T} \cdot e^{j\omega T} = |\ \ | \angle\ \ $

مقدار زاویه

$= r \angle \theta$

$= e^{\alpha T} \angle \omega T \Rightarrow T = 1sec \Rightarrow r = e^{\alpha},\ \theta = \omega$



for $S_0 = 0 + j0 \Rightarrow r = e^{0} \angle 0 = 1 \angle 0$

" $S_1 = j\pi/4 \Rightarrow r = e^{0} \angle \pi/4 = 1 \angle \pi/4 = 45^{\circ}$

" $S_2 = j\pi/2 \Rightarrow r = e^{0} \angle \pi/2 = 1 \angle 90^{\circ}$

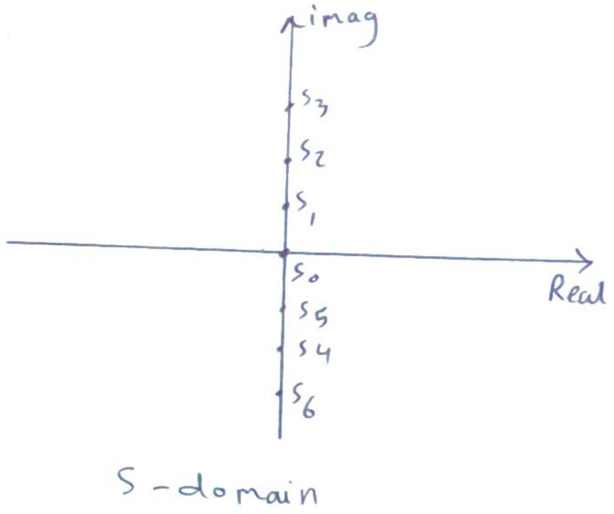
" $S_3 = j\pi \Rightarrow r = e^{0} \angle \pi = 1 \angle 180$

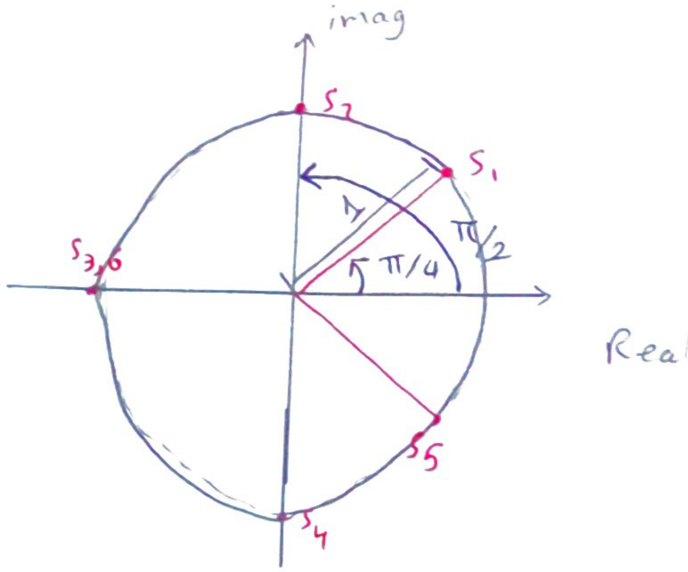
" $S_4 = -j\pi/2 \Rightarrow r = e^{0} \angle -\pi/2 = 1 \angle -90$

" $S_5 = -j\pi/4 \Rightarrow r = e^{0} \angle -\pi/4 = 1 \angle -45$

" $S_6 = -j\pi \Rightarrow r = e^{0} \angle -\pi = 1 \angle -180$

⟹ Continue





Mapping the imag. axis from S-domain to Z-domain is a circle with radius $r = 1$ and center at origin

- for LHS in S-domain ($S = \alpha + j\omega$)

  e^{α} ($\alpha < 0$ (-ve))

  $e^{\alpha} < 1 \Rightarrow |Z| = r = |e^{\alpha}| < 1$ (Stable)

  (Pole lies in unit circle; stable pole)

- for RHS in S-domain ($S = \alpha + j\omega$)

  e^{α} ($\alpha > 0$ (+ve))

  $e^{\alpha} > 1 \Rightarrow |Z| = r = |e^{\alpha}| > 1$ (unstable)

  (Pole lies outside of unit circle; unstable pole)

- For imag axis

  $\alpha = 0 \Rightarrow e^{\alpha = 0} = 1$ (critical Pole)

  $Z = r = e^{\alpha} = 1$ (critical Pole)

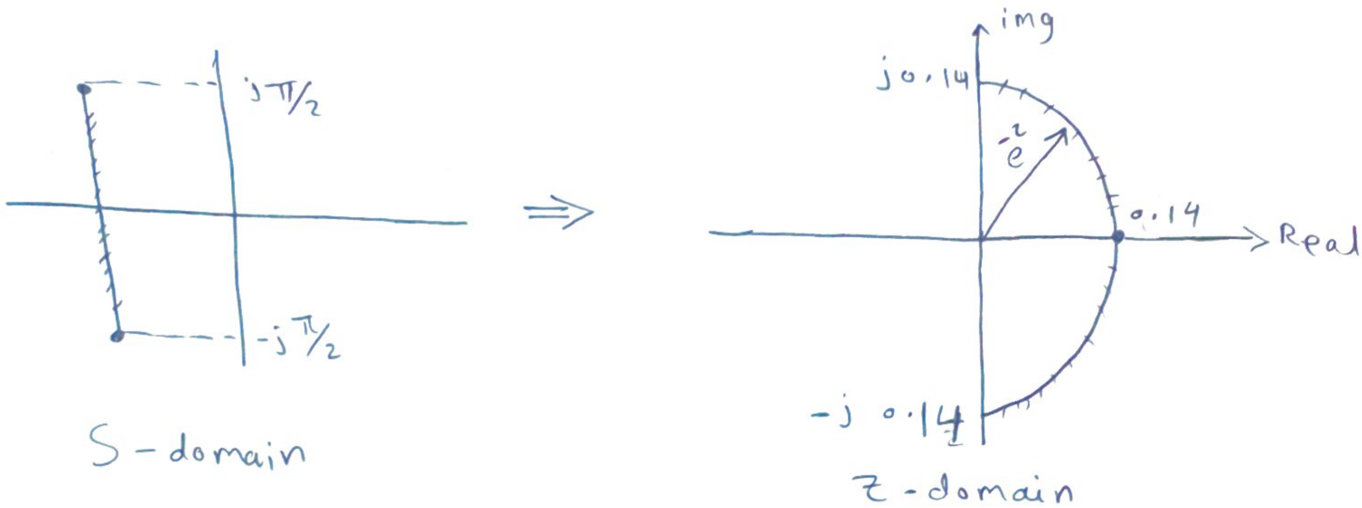
  (Pole lies on unit circle; critical pole)

1. The System is stable if all poles lie inside unit circle ($|z_p| < 1$)

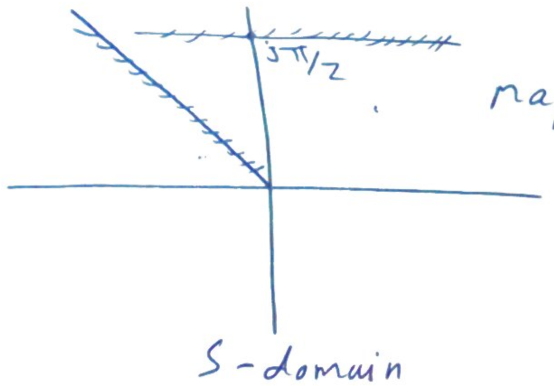
2. The system is unstable if at least one pole lies outside unit circle ($|z| > 1$)

3. The system is Critical stable, if one or more pole lie on the unit circle and the other poles lie inside the unit circle

Example: map to z-domain



Report



map to z-domain